4206. *Proposed by Gheorghe Alexe and George-Florin Serban.*

Find positive integers p and q that are relatively prime to each other such that $p + p^2 = q + q^2 + 3q^3$.

We received 19 complete solutions. We present the one by Prithwijit De.

We observe that $p+p^2$ is even for any positive integer p. Therefore in any solution q must be even. By rewriting the given equation as

$$p(1 + p) = q(1 + q + 3q^2)$$

we obtain $p|(1 + q + 3q^2)$ and $q|(p + 1)$. We may also rewrite the equation as

$$(p - q)(p + q + 1) = 3q^3$$

which implies $p > q$. Since $\gcd(p - q, q) = \gcd(p, q) = 1$, we can conclude that $q^3|(p + q + 1)$ and therefore

$$q^3 - q - 1 \leq p \leq 1 + q + 3q^2,$$

which leads to

$$q^3 - 3q^2 - 2q - 2 \leq 0.$$

Thus $q \leq 3$ and since q is positive and even, $q = 2$. We obtain $(p, q) = (5, 2)$ as the only solution.

4207. *Proposed by Mihaela Berindeanu.*

Let x, y and z be real numbers such that $x + y + z = 3$. Show that

$$\frac{1}{1 + 2^{4-3x}} + \frac{1}{1 + 2^{4-3y}} + \frac{1}{1 + 2^{4-3z}} \geq 1.$$

We received 18 solutions. We present 2 solutions.

Solution 1, by AN-anduud Problem Solving Group.

We have $2^{4-3x} \cdot 2^{4-3y} \cdot 2^{4-3z} = 8$, hence there exist a, b, c positive real numbers satisfying the following equalities:

$$2^{4-3x} = \frac{2ab}{c^2}, \qquad 2^{4-3y} = \frac{2bc}{a^2}, \qquad 2^{4-3z} = \frac{2ca}{b^2}.$$

The given inequality is equivalent to

$$\frac{1}{1 + \frac{2ab}{c^2}} + \frac{1}{1 + \frac{2bc}{a^2}} + \frac{1}{1 + \frac{2ca}{b^2}} \geq 1$$

$$\iff \frac{c^2}{c^2 + 2ab} + \frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} \geq 1 \qquad (1)$$

Using Cauchy-Schwarz inequality, we get

$$\frac{c^2}{c^2+2ab}+\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}\geq\frac{(a+b+c)^2}{(c^2+2ab)+(a^2+2bc)+(b^2+2ca)}=1.$$

Thus inequality (1) is proved. Equality holds if and only if $x=y=z=1$.

Solution 2, by Arkady Alt.

Let $a=2^{4-3x}$, $b=2^{4-3y}$, $c=2^{4-3y}$. Then $a,b,c>0$ and

$$abc=2^{12-3(x+y+z)}=8.$$

The original inequality becomes

$$\sum_{cyc}\frac{1}{1+a}\geq 1\quad\Longleftrightarrow\quad\sum_{cyc}(1+b)(1+c)\geq(1+a)(1+b)(1+c)$$

The last inequality gives

$$\begin{aligned}3+2(a+b+c)+ab+bc+ca &\geq 1+a+b+c+ab+bc+ca+abc\\ &= 9+a+b+c+ab+bc+ca,\end{aligned}$$

so $a+b+c\geq 6$, which is true because by AM-GM Inequality

$$a+b+c\geq 3\sqrt[3]{abc}=3\sqrt[3]{8}=6.$$

4208. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x,y and z be positive real numbers such that $x\leq y\leq z$. Prove that for any real number $k>2$, we have:

$$xy^k+yz^k+zx^k\geq x^2y^{k-1}+y^2z^{k-1}+z^2x^{k-1}.$$

We received 8 solutions. We present the one by Digby Smith.

Since $0<x\leq y\leq z$ and $k>2$, we have $0<x^{k-2}\leq y^{k-2}\leq z^{k-2}$. Thus

$$\begin{aligned}&(xy^k+yz^k+zx^k)-(x^2y^{k-1}+y^2z^{k-1}+z^2x^{k-1})\\ &=xy(y-x)y^{k-2}+yz(z-y)z^{k-2}+zx(x-z)x^{k-2}\\ &\geq xy(y-x)x^{k-2}+yz(z-y)x^{k-2}+zx(x-z)x^{k-2}\\ &=(xy^2-x^2y+yz^2-y^2z+zx^2-xz^2)x^{k-2}\\ &=(z-y)(y-x)(z-x)x^{k-2},\end{aligned}$$

where the last line is clearly non-negative. Hence the desired inequality follows, and clearly equality holds if and only if $x=y=z$.